Review Session:

1. Ito lemma:

If x follows Ito process, $dx = a(x,t)dt + b(x,t)dz$, dz is Brownian motion,

G is function of x and t, then $dG = (\frac{\partial G}{\partial x}a + \frac{\partial G}{\partial t} + \frac{1}{2}\cdot\frac{\partial^2 G}{\partial x^2}b^2)dt + \frac{\partial G}{\partial x}b\,dz$.

Example: S follows a geometric Brownian motion with drift μ and volatility σ

Show that the process $V = \sqrt{S}e^{2t}$ follows a geometric Brownian motion.

What are the drift and standard deviation of V.

By Ito Lemma, $dV = \frac{\partial V}{\partial S}dS + \frac{\partial V}{\partial t}dt + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}\cdot dSdS$

where $\frac{\partial V}{\partial S} = \frac{1}{2\sqrt{S}}e^{2t}$, $\frac{\partial^2 V}{\partial S^2} = -\frac{1}{4}\cdot\frac{1}{S^{3/2}}e^{2t}$, $\frac{\partial V}{\partial t} = 2\sqrt{S}e^{2t}$.

$dS\cdot dS = \sigma^2 S^2 dt$

Then $dV = \frac{1}{2\sqrt{S}}e^{2t}(\mu S dt + \sigma S dz) + 2\sqrt{S}e^{2t}dt + \frac{1}{2}\sigma^2 S^2(-\frac{1}{4})\frac{e^{2t}}{S^{3/2}}dt$

$= (\frac{\mu}{2} + 2 - \frac{\sigma^2}{8})V dt + \frac{1}{2}\sigma V dz$.

the drift is $\frac{\mu}{2} + 2 - \frac{\sigma^2}{8}$, the volatility is $\frac{\sigma}{2}$.

2. Correlated Geometric Brownian Motion.

X_1, X_2 are ~~two~~ stock prices follow GBM, $\frac{dX_1}{X_1} = a_1 dt + b_1 dz_1$, $\frac{dX_2}{X_2} = a_2 dt + b_2 dz_2$.

z_1 and z_2 are correlated. For two independent Brownian motion W_1, W_2,

$z_1 = W_1$, $z_2 = \rho W_1 + \sqrt{1-\rho^2}\,W_2$. then the log-returns of the two stocks are correlated with correlation ρ.

Example: Two stocks follow correlated Geometric Brownian motion with a correlation of 0.6. They both have a drift of $\mu_1 = \mu_2 = 0.08$. Stock 1 has a current price of \$10 and volatility of 0.3. Stock 2 has a current price of \$40 and volatility of 0.4. If at the end of 3 months, stock 1 finishes at \$12. What is the probability that stock 2 is less than \$40?

$$S_1(T) = S_1(0)\, e^{(\mu_1 - \frac{\sigma_1^2}{2})T} \cdot e^{\sigma_1\sqrt{T}\cdot \varepsilon_1}$$ and

$$S_2(T) = S_2(0) \cdot e^{(\mu_2 - \frac{\sigma_2^2}{2})T} \cdot e^{\sigma_2\sqrt{T}\cdot(\rho\varepsilon_1 + \sqrt{1-\rho^2}\,\varepsilon_2)}$$

$\varepsilon_1, \varepsilon_2$ are independent standard normal.

Since $S_1(T) = 12$, $S_1(0) = 10$, $\mu = 0.08$, $\sigma_1 = 0.3$, $T = \frac{1}{4}$.

then $$W_1(T) = \frac{\ln(\frac{12}{10}) - (0.08 - \frac{0.3^2}{2})\frac{1}{4}}{0.3\sqrt{\frac{1}{4}}} \approx 1.157.$$

For the second stock, $P(40 \cdot e^{(0.08 - \frac{0.4^2}{2})\frac{1}{4}} \cdot e^{0.4\sqrt{\frac{1}{4}}(0.6 \cdot 1.157 + \sqrt{1-0.6^2}\cdot \varepsilon_2)} < 40)$

$$= P(\varepsilon_2 < -\frac{0.6 \times 1.157}{0.8}) \approx P(\varepsilon_2 < -0.868)$$

$$= 0.193.$$

3. Black–Scholes Formula.

Call option: $$C = N(d_1)S_t - N(d_2)Ke^{-r(T-t)}, \quad d_1 = \frac{1}{\sigma\sqrt{T-t}}\left(\ln\left(\frac{S_t}{K}\right) + (r + \frac{\sigma^2}{2})(T-t)\right)$$

$$d_2 = d_1 - \sigma\sqrt{T-t}$$

Put option: $$P = N(-d_2)Ke^{-r(T-t)} - N(-d_1)S_t.$$

Example: An European option has a price given by $SN(d_1) - SN(d_3) - 35e^{-r\tau}N(d_2) + 45e^{-r\tau}N(d_4)$.

where $d_1 = \dfrac{\ln(\frac{S_0}{35}) + (r + \frac{\sigma^2}{2})\tau}{\sigma\sqrt{\tau}}$, $d_2 = d_1 - \sigma\sqrt{\tau}$. $d_3 = \dfrac{\ln(\frac{S_0}{45}) + (r + \frac{\sigma^2}{2})\tau}{\sigma\sqrt{\tau}}$

$d_4 = d_3 - \sigma\sqrt{\tau}$. and $\tau = 1$ year. What is the payoff for the option?

$V_T = (S_T - 35)^+ - (S_T - 45)^+$. Long a call at 35 while short a call at 45.